The canonical formalism and path integrals in curved spaces

1988 J. Phys. A: Math. Gen. 21 3355

(http://iopscience.iop.org/0305-4470/21/16/012)

J. Phys. A: Math. Gen. 21 (1988) 3355-3363. Printed in the UK

The canonical formalism and path integrals in curved spaces

J C D'Olivo† and M Torres‡

† Centro de Estudios Nucleares, Universidad Nacional Autonóma de México, Circuito Exterior CU, 04510 México, DF

‡ Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 México, DF

Received 15 December 1987, in final form 16 March 1988

Abstract. We discuss the canonical and path integral quantisation of a particle moving on a curved manifold. We advance a simple prescription to quantise the system within the canonical formalism. An effective Hamiltonian H_{eff} is defined as the classical Hamiltonian plus a quantum potential term of order $\hbar^2$. The Hamilton operator is obtained by Weyl ordering the expression that results when one replaces coordinates and momenta by their corresponding Hermitian operators in the effective Hamiltonian. We construct the phase space path integral representation for the propagator in terms of H_{eff}. When the propagator is expressed as a Lagrangian path integral with an invariant measure, an additional correction is introduced.

1. Introduction

In this paper we consider the description of a quantum mechanical system in a curved manifold. In particular, we try to emphasise some relations between the path integral and the canonical formalism.

The formulation of path integrals on curved manifolds has been considered by DeWitt [1] and several other authors [2-8]. The main conclusion that is drawn from this work is that a careful treatment leads to an effective action constructed in terms of an effective Lagrangian

$$L_{\text{eff}} = L - \Delta V \tag{1.1}$$

where L represents the original Lagrangian and ΔV is a quantum correction proportional to $\hbar^2$. When used in the path integral this effective Lagrangian (and not the classical one) gives the correct expression for the Schrödinger propagator.

On the other hand, it is well known that, in the framework of canonical quantisation, the ordering of non-commuting variables leads to serious difficulties. For systems described by curvilinear coordinates, these kinds of difficulties were recognised almost since the beginnings of quantum mechanics [9]. To deal with them, several ordering rules have been proposed, e.g. the rule of Born-Jordan, the Weyl rule, the symmetrisation rule, etc. Here, we consider that, for a particle moving in a curved manifold M, the kinetic part of the Hamiltonian operator $\hat{H}$ coincides with the covariant Laplacian defined on M. The replacement of coordinates and momenta by their corresponding operators in the classical Hamiltonian accompanied by the choice of the Weyl ordering gives us an operator $[H(\hat{q}, \hat{p})]_{\mathrm{w}}$ that differs from $\hat{H}$ in a quantity ΔV. As we show, this quantity is related to the correction that appears in the effective Lagrangian used in the path integrals.

The organisation of the paper is as follows. In § 2 we derive a general expression for the quantum correction. The phase space path integral representation of the Schrödinger propagator is derived in § 3. The calculations are done using the results obtained in § 2 and adopting the midpoint prescription. After performing a momentum integration we express the result in terms of a Feynman integral over paths in the configuration space. Finally, we compare our results with previous calculations.

2. The Effective Hamiltonian in curved spaces

Consider an n-dimensional manifold M with coordinates $q^i (i=1, \ldots, n)$ and a metric g_{ij} defined on it. We assume that the manifold is non-compact, with each q^i ranging from $-\infty$ to ∞. The Lagrangian for a particle moving on M is

$$L(q, \dot{q}) = \tfrac{1}{2}\dot{q}^i g_{ij}\dot{q}^j + A_i\dot{q}^i - V \tag{2.1}$$

where dots represent differentiation with respect to the time t and the summation convention is adopted. In three dimensions, A_i and V can be considered as the vector and scalar electromagnetic potentials respectively, but in general they are functions of q and t without such special meaning. In the following we assume that they only depend on the coordinates and similarly for g_{ij}. The momenta and the Hamiltonian function associated with L are given by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j + A_i \tag{2.2}$$

and

$$H(q, p) = p_i\dot{q}^i - L = \tfrac{1}{2}(q_i - A_i)g^{ij}(p_j - A_j) + V \tag{2.3}$$

respectively, where g^{ij} is the inverse of g_{ij} ($g^{ij}g_{jk} = \delta^i_k$).

When we move from classical to quantum mechanics the canonical variables q and p become Hermitian operators, $\hat{q}$ and $\hat{p}$, with eigenstates $|q, t\rangle$ and $|p, t\rangle$. These operators satisfy the commutation relations

$$[\hat{q}^i(t), \hat{q}^j(t)] = [\hat{p}_i(t), \hat{p}_j(t)] = 0 \qquad [\hat{q}^i(t), \hat{p}_j(t)] = \mathrm{i}\hbar\delta^i_j. \tag{2.4}$$

The eigenstates of $\hat{q}^i$ are normalised according to

$$\langle q'', t|q', t\rangle = \delta(q''; q') = g^{-1/4}(q'')\delta(q''-q')g^{-1/4}(q'). \tag{2.5}$$

The quantity $g(q)$ is the determinant of the metric tensor and $\delta(q''-q')$ is an ordinary n-dimensional delta function.

We assume that the eigenstates $|q, t\rangle$ form a complete set

$$\int g^{1/2}(q)\,\mathrm{d}^n q|q, t\rangle\langle q, t| = 1 \tag{2.6}$$

where $g^{1/2}(q)\,\mathrm{d}^n q = g^{1/2}(q)\,\mathrm{d}q^1 \ldots \mathrm{d}q^n$ represents the invariant volume element.

The wavefunction $\Psi(q, t) = \langle q, t|\Psi\rangle$ associated with the state $|\Psi\rangle$ obeys the Schrödinger equation

$$\mathrm{i}\hbar\partial\Psi(q, t)/\partial t = \hat{H}\Psi(q, t) \tag{2.7}$$

where the Hermitian operator $\hat{H}$ represents the Hamiltonian of the system. For quantised systems with no non-classical degrees of freedom like spin Ψ is a scalar function.

The Hamilton operator has to be constructed in such a way as to guarantee the covariance of the wave equation. For the special case of a flat space, following Podolsky [9], one can first write down the Hamiltonian in cartesian coordinates and then carry out the change of coordinates. The result is

$$\hat{H} = -\tfrac{1}{2}\hbar^2\Delta_2 + \mathrm{i}\hbar A^i\partial_i + \mathrm{i}\tfrac{1}{2}\hbar g^{-1/2}\partial_i(g^{1/2}A^i) + \tfrac{1}{2}A^iA_i + V \tag{2.8}$$

where $\partial_i = \partial/\partial q^i$ and Δ_2 represents the covariant Laplace-Beltrami operator

$$\Delta_2 = g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j). \tag{2.9}$$

We will assume that equation (2.8) remains valid even for non-vanishing curvature. The quantum operator associated with the classical Lagrangian (2.1) could, in principle, contain additional (scalar) contributions constructed from the Ricci tensor. All these possible extra terms vanish when $\hbar \to 0$ and therefore they do not contribute to the classical limit. For simplicity we do not include them.

The commutation relations (2.4), together with the hermicity condition, give us the following coordinate representation for the momentum operator:

$$\hat{p}_i = -\mathrm{i}\hbar(\partial_i + \tfrac{1}{2}\Gamma_i) \tag{2.10}$$

with

$$\Gamma_i = \Gamma^l_{li} = g^{-1/2}\partial_i g^{1/2} \tag{2.11}$$

Γ^k_{ij} being the affine connection or Christofell symbol of the second kind ($\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$).

In the present paper, we adopt the Weyl rule [10] for constructing the quantum operator corresponding to the classical Hamiltonian (2.3). In this way, we will be able to derive an explicit expression for the quantum potential.

For our purpose, we only need to consider functions which depend arbitrarily on q, but at most quadratically on p. In this case, the corresponding Weyl-ordered operators are given by

$$[f(\hat{q})\hat{p}]_{\mathrm{w}} = \tfrac{1}{2}(f(\hat{q})\hat{p} + \hat{p}f(\hat{q})) \tag{2.12}$$

$$[f(\hat{q})\hat{p}^2]_{\mathrm{w}} = \tfrac{1}{4}(f(\hat{q})\hat{p}^2 + 2\hat{p}f(\hat{q})\hat{p} + \hat{p}^2 f(\hat{q})). \tag{2.13}$$

Equation (2.13), together with (2.10), gives us

$$\begin{aligned}[\hat{p}_i g^{ij}(\hat{q})\hat{p}_j]_{\mathrm{w}} &= \tfrac{1}{4}(\hat{p}_i\hat{p}_j g^{ij} + 2\hat{p}_i g^{ij}\hat{p}_j + g^{ij}\hat{p}_i\hat{p}_j)\\ &= -\tfrac{1}{4}\hbar^2\{4g^{ij}[\partial_i\partial_j + \Gamma_i\partial_j + \tfrac{1}{2}(\partial_i\Gamma_j) + \tfrac{1}{4}\Gamma_i\Gamma_j] + 4(\partial_i g^{ij})(\partial_j + \tfrac{1}{2}\Gamma_j) + (\partial_i\partial_j g^{ij})\}.\end{aligned} \tag{2.14}$$

After some algebra this expression can be written as

$$[\hat{p}_i g^{ij}(\hat{q})\hat{p}_j]_{\mathrm{w}} = -\hbar^2(\Delta_2 + \tfrac{1}{4}R + \tfrac{1}{4}g^{ij}\Gamma^k_{il}\Gamma^l_{jk}) \tag{2.15}$$

where Δ_2 is the second-order differential operator defined in equation (2.9) and $R = g^{ij}R_{ij}$ represents the scalar curvature. In our notation [11] the Ricci tensor $R_{ij} = R^k{}_{ijk}$ is given by

$$R_{ij} = \partial_j\Gamma^k_{ik} - \partial_k\Gamma^k_{ij} + \Gamma^l_{ik}\Gamma^k_{jl} - \Gamma^l_{ij}\Gamma^k_{kl}. \tag{2.16}$$

Next, we consider the other terms of (2.3). From (2.12) we immediately see that

$$\begin{aligned}[g^{ij}(\hat{q})A_i(\hat{q})\hat{p}_j]_{\mathrm{w}} &= \tfrac{1}{2}(g^{ij}A_i\hat{p}_j + \hat{p}_j g^{ij}A_i)\\ &= -\mathrm{i}\hbar(A^i\partial_i + \tfrac{1}{2}g^{-1/2}\partial_i(g^{1/2}A^i)).\end{aligned} \tag{2.17}$$

Finally, using the results of equations (2.15) and (2.17) and taking into account that $[A^iA_i]_w = A^iA_i$ and $[V]_w = V$ we find

$$\hat{H}_w = \hat{H} - \Delta V. \tag{2.18}$$

Here $\hat{H}$ is the Hamiltonian given in (2.8) which we are presently assuming to describe the system, while

$$\Delta V(q) = \tfrac{1}{8}\hbar^2(R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk}) \tag{2.19}$$

represents a quantum correction which vanishes in the classical limit ($\hbar \to 0$).

The foregoing results can be used to outline a procedure to quantise the system along the usual canonical formalism as follows.

Corresponding to the Lagrangian (2.1) that describes the classical dynamics of a particle moving in a Riemannian space, we associate an effective Lagrangian L_{eff} given by

$$L_{\text{eff}}(q, \dot{q}) = L(q, \dot{q}) - \Delta V(q) = \tfrac{1}{2}\dot{q}^i g_{ij}\dot{q}^j + A_i\dot{q}^i - (V + \Delta V) \tag{2.20}$$

where ΔV is determined by the structure of the Riemann manifold and is given by (2.19). From (2.20) and (2.2) we can calculate the effective Hamiltonian

$$H_{\text{eff}}(q, p) = \dot{q}^i p_i - L_{\text{eff}}(q, p) = \tfrac{1}{2}(p_i - A_i)g^{ij}(p_j - A_j) + V + \Delta V. \tag{2.21}$$

The system is now quantised, replacing the variables q and p by Hermitian operators $\hat{q}$ and $\hat{p}$ that fulfil the commutation relations (2.4). The Hamiltonian operator $\hat{H}$ of the system is obtained when one makes the above replacement in $H_{\text{eff}}(q, p)$ and uses the Weyl-ordering rule, i.e.

$$\hat{H} = [H_{\text{eff}}(\hat{q}, \hat{p})]_w = \tfrac{1}{2}[\hat{p}_i g^{ij}\hat{p}_j]_w - [A_i g^{ij}\hat{p}_j]_w + \tfrac{1}{2}A^iA_i + V + \Delta V. \tag{2.22}$$

By using the coordinate realisation of $\hat{p}$ as given in (2.10), the right-hand side of (2.22) reduces to (2.8).

It is important to remark on the following points. (i) The effective Lagrangian and the effective Hamiltonian are classical in the sense that they depend on classical variables q and p, but they contain an extra potential term that is quadratic in the Planck constant. (ii) When the Hamiltonian operator $\hat{H}$ is written as a Weyl-ordered expression it shows an explicit dependence on the effective quantum potential. In the next section we will see that the result (2.22) is particularly suited to derive the path integral representation for the propagator.

3. From the canonical to the path integral formalism

The propagator from q' at time t' to q'' at time t'' is

$$\mathbb{K}(q'', t''|q', t') = \langle q''| \exp[-(\mathrm{i}/\hbar)(t'' - t')\hat{H}]|q'\rangle \tag{3.1}$$

where $\hat{H}$ is the time-independent Hamilton operator given in (2.8) that, according to (2.22), coincides with $[\hat{H}_{\text{eff}}]_w$.

We follow the standard procedure to obtain the path integral representation for (3.1), namely, we split up the time interval (t', t'') into a large number N of short intervals. Then we use the completeness relation (2.6) to write

$$\mathbb{K}(q'', t''|q', t) = \lim_{N\to\infty} \int \prod_{a=1}^{N-1} [g(q_a)]^{1/2}\, \mathrm{d}^n q_a \prod_{b=1}^{N} \langle q_b| \exp[-(\mathrm{i}/\hbar)\varepsilon[\hat{H}_{\text{eff}}]_w]|q_{b-1}\rangle \tag{3.2}$$

where $\varepsilon = (t'' - t')/N$ and $q_0 = q'$, $q_N = q''$.

The eigenvectors of the momentum operator (2.10) are chosen to have the coordinate representation

$$\langle q|p\rangle = \frac{1}{(2\pi\hbar)^{n/2}}\left(\frac{1}{u(p)g(q)}\right)^{1/4}\exp\left(\frac{\mathrm{i}}{\hbar}p.q\right) \tag{3.3}$$

where $p.q = p_i q^i$. The arbitrary function $u(p)$ is included in order to identify $[u(p)]^{1/2}\,\mathrm{d}^n p$ with the volume element in momentum space. The expectation values of $\hat{p}^r$ $(r=1,2)$ are calculated using the completeness relation in momentum space and (3.3):

$$\begin{aligned}\langle q'|\hat{p}_i^r|q\rangle &= \int [u(p)]^{1/2}\,\mathrm{d}^n p\langle q'|\hat{p}_i^r|p\rangle\langle p|q\rangle \\ &= \frac{1}{[g(q)g(q')]^{1/4}}\int\frac{\mathrm{d}^n p}{(2\pi\hbar)^n}p_i^r\exp\left(\frac{\mathrm{i}}{\hbar}p.(q'-q)\right).\end{aligned} \tag{3.4}$$

Notice that $u(p)$ has cancelled out in the final expression.

We need to compute the expectation value for a Weyl-ordered product of momentum and position operators. For $f(\hat{q}) = \hat{q}^m$ equations (2.12) and (2.13) may be written as

$$[\hat{q}^m\hat{p}^r]_{\mathrm{w}} = \left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}\hat{q}^{m-l}\hat{p}^r\hat{q}^l \tag{3.5}$$

that is used to obtain

$$\langle q'|[\hat{q}^m\hat{p}^r]_{\mathrm{w}}|q\rangle = \frac{[(q+q')/2]^m}{[g(q)g(q')]^{1/4}}\int\frac{\mathrm{d}^n p}{(2\pi\hbar)^n}p^r\exp\left(\frac{\mathrm{i}}{\hbar}p.(q'-q)\right). \tag{3.6}$$

In (3.5) and (3.6), $r=1,2$. For a product of $\hat{p}$ or $\hat{p}^2$ with an arbitrary function of $f(\hat{q})$, we expand $f(\hat{q})$ and use (3.4) and (3.6) to get

$$\langle q'|[f(\hat{q})\hat{p}^r]_{\mathrm{w}}|q\rangle = \frac{f[(q+q')/2]}{[g(q)g(q')]^{1/4}}\int\frac{\mathrm{d}^n p}{(2\pi\hbar)^n}p^r\exp\left(\frac{\mathrm{i}}{\hbar}p.(q'-q)\right). \tag{3.7}$$

We are now ready to compute the short-time propagator that appears in equation (3.2). Employing the result given in (3.7) together with (2.22) and the orthogonal relation (2.5), to order ε we obtain

$$\begin{aligned}&\langle q_b|\exp(-\mathrm{i}\hbar^{-1}\varepsilon[\hat{H}_{\mathrm{eff}}]_{\mathrm{w}})|q_{b-1}\rangle \\ &\quad\simeq\frac{1}{[g(q_b)g(q_{b-1})]^{1/4}}\int\frac{\mathrm{d}^n p}{(2\pi\hbar)^n} \\ &\qquad\times\{1-\mathrm{i}\hbar^{-1}\varepsilon[\tfrac{1}{2}p_i g^{ij}(\bar{q}_b)p_j - A_i(\bar{q}_b)g^{ij}(\bar{q}_b)p_j \\ &\qquad+\tfrac{1}{2}A_i(\bar{q}_b)A^i(\bar{q}_b)+V(\bar{q}_b)+\Delta V(\bar{q}_b)]\} \\ &\qquad\times\exp\left(\frac{\mathrm{i}}{\hbar}p.(q_b-q_{b-1})\right) \\ &\quad\simeq\frac{1}{[g(q_b)g(q_{b-1})]^{1/4}}\int\frac{\mathrm{d}^n p}{(2\pi\hbar)^n}\exp\left(\frac{\mathrm{i}}{\hbar}\varepsilon[p.\Delta q_b/\varepsilon - H_{\mathrm{eff}}(\bar{q}_b,p)]\right)\end{aligned} \tag{3.8}$$

where $\bar{q}_b = (q_b+q_{b-1})/2$ and $\Delta q_b^i = (q_b-q_{b-1})^i$. Notice that the infinitesimal time propagator has been expressed as a function of the effective Hamiltonian given in (2.21) evaluated at the midpoint $\bar{q}_b$. Substituting the above expression into (3.2) we obtain the following expression for the complete propagator:

$$\mathbb{K}(q'', t''|q', t') = \lim_{N\to\infty} \frac{1}{[g(q')g(q'')]^{1/4}} \int \prod_{a=1}^{N-1} \mathrm{d}^n q_a \prod_{b=1}^{N} \frac{\mathrm{d}^n p_b}{(2\pi\hbar)^n}$$
$$\times \exp\left(\frac{\mathrm{i}}{\hbar} \varepsilon \sum_{b=1}^{N} [p_b.\Delta q_b/\varepsilon - H_{\mathrm{eff}}(\bar{q}_b, p_b)]\right). \tag{3.9}$$

Note that p and q are classical variables. Problems with operator ordering have apparently been lost, but they are really translated to the choice of the specific point where the functions are evaluated [12-14].

It is interesting to remark that equation (3.9) looks like a usual Hamiltonian path integral representation for the propagator; however, an effective Hamiltonian has to be used. This effective Hamiltonian has an extra contribution

$$\Delta V = \tfrac{1}{8}\hbar^2 (R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk})$$

compared to the classical Hamiltonian and takes into account that we are working in a curved manifold.

In order to cast equation (3.9) into a Lagrangian path integral over configuration space, we have to perform the momentum integrations. This can be readily carried out using

$$\int \frac{\mathrm{d}^n p}{(2\pi)^{n/2}} \exp\left(-\frac{\mathrm{i}}{\hbar} \varepsilon [\tfrac{1}{2}(p_i - A_i) g^{ij} (p_j - A_j) - p_i \Delta q^i/\varepsilon - (V + \Delta V)]\right)$$
$$= \left(\frac{\hbar}{\mathrm{i}\varepsilon}\right)^{n/2} g^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar} \varepsilon \left(\frac{1}{2} \frac{\Delta q^i}{\varepsilon} g_{ij} \frac{\Delta q^j}{\varepsilon} + A_i \frac{\Delta q^i}{\varepsilon} - (V + \Delta V)\right)\right] \tag{3.10}$$

where all the functions are evaluated at the midpoint $(q+q')/2$ and $\Delta q^i = (q'-q)^i$. The result is

$$\mathbb{K}(q'', t''|q', t') = \lim_{N\to\infty} \int \left(\frac{1}{2\pi\mathrm{i}\hbar\varepsilon}\right)^{nN/2} \prod_{a=1}^{N-1} g^{1/2}(q_a)\, \mathrm{d}^n q_a$$
$$\times \prod_{b=1}^{N} \frac{[g(\bar{q}_b)]^{1/2}}{[g(q_b)g(q_{b-1})]^{1/4}} \exp\left[\frac{\mathrm{i}}{\hbar} \varepsilon L_{\mathrm{eff}}\left(\bar{q}_b, \frac{\Delta q_b}{\varepsilon}\right)\right] \tag{3.11}$$

$L_{\mathrm{eff}}(q, \dot{q})$ being the effective Lagrangian defined in (2.20) evaluated at $q = \bar{q}_b$ and $\dot{q} = (q_b - q_{b-1})/\varepsilon$.

It seems that there is an additional contribution to the measure coming from the factor $[g(\bar{q}_b)]^{1/2}/[g(q_b)g(q_{b-1})]^{1/4}$, which will produce a non-invariant volume element. However, it is possible to incorporate it into the exponential factor as an extra potential term. We need the result

$$[g(q_b)g(q_{b-1})]^{1/4} = [g(\bar{q}_b)]^{1/2}(1 + \tfrac{1}{8}\partial_i \Gamma^l_{lj}(\bar{q}_b)\Delta q^i_b \Delta q^j_b + \dots]. \tag{3.12}$$

In (3.12) we have only retained terms that are quadratic in Δq because the higher-order terms do not contribute to the path integral in the limit $\varepsilon \to 0$. This is true since terms of order Δq^2 act like terms of order ε when in the integrand of a path integral. In fact, under a path integral it is valid to make the following replacement [1, 2]:

$$\Delta q^i \Delta q^j \to \mathrm{i}\hbar\varepsilon g^{ij}. \tag{3.13}$$

Substituting (3.12) into (3.11) and using (3.13) we obtain the Lagrangian path integral representation for the propagator

$$\mathbb{K}(q'',t''|q',t')=\lim_{N\to\infty}\int\left(\frac{1}{2\pi \mathrm{i}\hbar\varepsilon}\right)^{nN/2}\prod_{a=1}^{N-1}g^{1/2}(q_a)\,\mathrm{d}^n q_a$$
$$\times\prod_{b=1}^{N}\exp\left(\frac{\mathrm{i}}{\hbar}\,\varepsilon[L_{\mathrm{eff}}(\bar q_b,\Delta q_b/\varepsilon)-\Delta V'(\bar q_b)]\right) \tag{3.14}$$

where $\Delta V'=\frac{1}{8}\hbar^2 g^{ij}\partial_j\Gamma^l_{li}$. Notice that the measure contains the factor $g^{1/2}$ evaluated at the mesh points and therefore the volume element $g^{1/2}(q_a)\,\mathrm{d}^n q_a$ is the invariant one for each a.

Equation (3.14) shows that, in order to represent the propagator by the Lagrangian form of the path integral, we have to add another correction $\Delta V'$ to the classical Lagrangian. So, the total quantum potential ΔV_{tot} becomes

$$\Delta V_{\mathrm{tot}}=\Delta V+\Delta V'=\tfrac{1}{8}\hbar^2[R+g^{ij}(\Gamma^k_{il}\Gamma^l_{jk}+\partial_j\Gamma^l_{li})]. \tag{3.15}$$

Equation (3.15) is essentially the result that was obtained by McLaughlin and Schulman [2], expanding the action around classical paths and retaining terms of order up to $(\Delta q)^4$ (see also [8]). Actually their result differs from (3.15) by a term $\frac{1}{12}\hbar^2 R$, but this is because we have computed the propagator generated by $\hat H$ (2.8), while in [1, 2, 8] they consider the propagator for the Hamiltonian $\hat H-\frac{1}{12}\hbar^2 R$. Considering the so-called Weyl transform Mizrahi [6] presented a derivation for the effective potential that contains the first two terms of the right-hand side of equation (3.15). As we have shown, an extra term appears if one insists in having an invariant measure. It is important to remark that the quantum potentials ΔV and $\Delta V'$ are present even in a flat space when non-cartesian coordinates are used. This question has been analysed by several authors [15–20] in connection with path integrals in curvilinear coordinates.

In a recent work [21], the quantum correction was derived using an ordering different from the Weyl one. ΔV was calculated according to†

$$\Delta V=\hat H-\tfrac{1}{2}[\hat p_i g^{ij}(q)\hat p_j]=\tfrac{1}{8}\hbar^2[g^{ij}\Gamma_i\Gamma_j+2\partial_i(g^{ij}\Gamma_j)] \tag{3.16}$$

with Γ_i given by (2.11). It is claimed, without a proof, that (3.16) is the 'correct' expression to be used in a Lagrangian path integral. Furthermore, Grosche and Steiner assert that other results for ΔV are wrong (e.g. DeWitt [1], McLaughlin and Schulman [2], Marinov [8], Gervais and Jevicki [19]). Apparently they did not take into account that different ordering rules in the Hamiltonian translate into different prescriptions to evaluate functions that appear in the path integrals [12–14]. Therefore, their claim is ill founded.

The ordering given in (3.16) corresponds to the prescription that, under the path integral, any function $f(q)$ has to be approximated by (see the appendix)

$$2f(\bar q_b)-\tfrac{1}{2}[f(q_b)+f(q_{b-1})] \tag{3.17}$$

where, as before, $\bar q_b$ represents the middle point. This fact has also been overlooked in their work. On the other hand, in the examples they presented, $f(q)$ was approximated by $f^{1/2}(q_b)f^{1/2}(q_{b-1})$ which, in general, is associated with an ordering different from (3.16) (see the appendix). However, it worked correctly since ΔV coincided for both orderings in the particular cases they considered.

† Without loss of generality, here we set V and A_i equal to zero.

4. Conclusions

In this paper we have considered the quantisation of a system in curved manifolds. The phase space path integral representation for the propagator was constructed in terms of an effective Hamiltonian which differs from the classical one by a quantum correction of order $\hbar^2$. A simple derivation of the general expression for this correction was first presented by adopting the canonical formalism. When the propagator is expressed as a Lagrangian path integral with an invariant measure, an additional contribution to the quantum potential is introduced.

Acknowledgment

The work of one of us (MT) has been supported by Consejo Nacional de Ciencia y Technología y Fondo Zevada (México).

Appendix

In this appendix, we list the expressions for the quantum potential ΔV corresponding to some ordering rules. We also state the prescription that should be used to evaluate any coordinate function $f(q)$ that appears in the Hamiltonian path integral. To derive these results we proceed in the same way as in § 2 of this paper.

(i) Weyl rule:

$$\Delta V_{\mathrm{w}} = \hat{K} - \tfrac{1}{2}[\hat{p}_i g^{ij}(q)\hat{p}_j]_{\mathrm{w}} = \tfrac{1}{8}\hbar^2[R + g^{ij}\Gamma^k_{il}\Gamma^l_{jk}]$$

where $\hat{K} = -\frac{1}{2}\hbar^2\Delta_2$.

Prescription: $f(q) \to f(\bar{q}_b)$.

(ii)

$$\Delta V = \hat{K} - \tfrac{1}{2}\hat{p}_i g^{ij}(q)\hat{p}_j = \Delta V_{\mathrm{w}} - \tfrac{1}{8}\hbar^2\partial_i\partial_j g^{ij}$$

$$= \tfrac{1}{8}\hbar^2[g^{ij}\Gamma_i\Gamma_j + 2\partial_i(g^{ij}\Gamma_j)].$$

Prescription: $f(q) \to 2f(\bar{q}_b) - \frac{1}{2}[f(q_b) + f(q_{b-1})]$.

(iii) Symmetric rule:

$$\Delta V_{\mathrm{s}} = \hat{K} - \tfrac{1}{4}[g^{ij}(q)\hat{p}_i\hat{p}_j + \hat{p}_i\hat{p}_j g^{ij}(q)] = \Delta V_{\mathrm{w}} + \tfrac{1}{8}\hbar^2\partial_i\partial_j g^{ij}.$$

Prescription: $f(q) \to \frac{1}{2}[f(q_b) + f(q_{b-1})]$.

(iv)

$$\Delta V = \hat{K} - \tfrac{1}{2}h^{ik}(q)\hat{p}_i\hat{p}_j h^{kj}(q)$$

$$= \Delta V_{\mathrm{w}} - \tfrac{1}{8}\hbar^2\partial_i\partial_j g^{ij} + \tfrac{1}{8}\hbar h^{ik}\partial_i\partial_j h^{kj}$$

where $h^{ik}h^{kj} = g^{ij}$.

Prescription: $f(q) \to [f(q_b)f(q_{b-1})]^{1/2}$.

In all the previous examples g^{ij} appears symmetrically with respect to $\hat{p}_i\hat{p}_j$ and this leads to a ΔV independent of $\hat{p}$. As we see below an asymmetric choice will produce an effective potential that is momentum dependent.

(v)
$$\begin{aligned}\Delta V &= \hat{K} - \tfrac{1}{2}g^{ij}(q)\hat{p}_i\hat{p}_j \\ &= \Delta V_{\rm w} - \tfrac{1}{8}\hbar^2\partial_i\partial_j g^{ij} - \tfrac{1}{2}\mathrm{i}\hbar(\partial_i g^{ij})\hat{p}_j.\end{aligned}$$

We recall that the transition from the Hamiltonian to the Lagrangian path integral can generate extra contributions to the effective potential.

Note added. After this work was completed we notice that, in the final version of the paper by Grosche and Steiner [22], they corrected some of the mistakes we have pointed out. Although they now compute ΔV using the Weyl ordering, they do not consider the third term of equation (3.15).

References

[1] DeWitt B S 1957 *Rev. Mod. Phys.* **29** 377
[2] McLaughlin D W and Schulman L S 1971 *J. Math. Phys.* **12** 2520
Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
[3] Cheng K S 1972 *J. Math. Phys.* **13** 1723
[4] Ben-Abraham S I and Lonke A 1973 *J. Math. Phys.* **14** 1935
Ringwood G A 1976 *J. Phys. A: Math. Gen.* **9** 1253
[5] Dowker J S 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1256
[6] Mizrahi M M 1975 *J. Math. Phys.* **16** 2201
[7] Omote M 1977 *Nucl. Phys.* B **120** 325
[8] Marinov M S 1980 *Phys. Rep.* **60** 1
[9] Podoslky B 1928 *Phys. Rev.* **32** 812
[10] Weyl H 1927 *Z. Phys.* **46** 1
Lee T D 1981 *Particle Physics and Introduction to Field Theory* (New York: Harwood)
[11] Weinberg W 1972 *Gravitation and Cosmology, Principles and Applications of the General Theory of Relativity* (New York: Wiley)
[12] Cohen L 1970 *J. Math. Phys.* **11** 3296
[13] Testa F J 1971 *J. Math. Phys.* **12** 1471
[14] Berezin F A 1971 *Theor. Math. Phys.* **6** 141
[15] Edwards S F and Gulyaev Y V 1964 *Proc. R. Soc.* A **279** 229
[16] Gutzwiller M C 1967 *J. Math. Phys.* **8** 1979
[17] Arthurs A M 1969 *Proc. R. Soc.* A **313** 445
[18] Peak D and Inomata A 1969 *J. Math. Phys.* **10** 1422
[19] Gervais J L and Jevicki A 1976 *Nucl. Phys.* B **110** 93
[20] Böhm M and Junker G 1987 *J. Math. Phys.* **28** 1978
[21] Grosche C and Steiner F 1987 *Preprint* DESY 87-014
[22] Grosche C and Steiner F 1987 *Z. Phys.* C **36** 699